Braid Group, Gauge Invariance, and Topological Order

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Topological order in two-dimensional systems is studied by combining the braid group formalism with a gauge invariance analysis. We show that flux insertions (or large gauge transformations) pertinent to the toroidal topology induce automorphisms of the braid group, giving rise to a unified algebraic structure that characterizes the ground-state subspace and fractionally charged, anyonic quasiparticles. Minimal ground-state degeneracy is derived without assuming any relation between quasiparticle charge and statistics. We also point out that noncommutativity between large gauge transformations is essential for the topological order in the fractional quantum Hall effect.

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In contrast with expectation from classical thermodynamics, physics at zero temperature is very rich because of the novel effects of quantum fluctuations. In recent years it has become increasingly clear that in a wide class of two-dimensional strongly correlated many-body systems, a transition driven by a nonthermal parameter may occur at zero temperature to a novel phase which cannot be described by usual spontaneous symmetry breaking and order parameters. The characteristic signature of the novel phase is a finite ground-state degeneracy that depends on the topology of the system; accompanying it are charge fractionalization (with respect to that of the constituent particles) and/or fractional statistics of the quasiparticles. The first known example is the Laughlin state [1] for the fractional quantum Hall (FQH) effect, with an electron filling factor $\nu = 1/n$ with n odd. Soon after, it was realized that in this phase the ground state is n-fold degenerate on a cylinder [2] or on a torus [3], while it is known to be nondegenerate on a sphere. Actually it is the ground-state degeneracy that is responsible for the fractional quantization of Hall conductance [2,4] and dictates the fractional charge $e^* = e/n$ [5] and the anyon statistics $\theta = \pi/n$ [3,6] of the quasiparticles. This type of new order is dubbed as topological order [7]. In recent years more systems, including bosonic ones or those at zero magnetic field, are identified as possessing topological order [8].

In the study of topological order, a central issue is how to characterize or classify topological orders. Previously there has been the idea [2,5] that the topology dependent ground-state degeneracy seems to be dictated by an (emergent) discrete symmetry. But the latter was never identified explicitly, except being Z_n for the Laughlin states. Another important issue is how to understand the relationship between ground-state degeneracy and charge fractionalization and/or quasiparticle statistics. A puzzling fact is that different patterns have appeared in investigations of various systems. For example, it was concluded [3] for the FQH systems that on a surface with nonzero genus g, the appearance of a fractional $\theta = \pi m/n$ statistics, with m and n coprimes, requires n^g-fold degenerate ground states, confirmed by the braid group analysis [6,9] and by the effective Chern-Simons theory as well [10,11]. On the other hand, in a recent paper [12] it was shown, by using a gauge invariance argument [5], that charge fractionalization with $e^* = ep/q$, with p and q coprimes, requires a ground-state degeneracy q^{2g} if the quasiparticles are ordinary bosons and fermions, while the FQH ground-state degeneracy is known to be only q^g-fold [3,9].

In this Letter, we start with a reexamination of the interplay between charge fractionalization and quasiparticle anyon statistics, if they coexist, in constraining ground-state degeneracy. This has been studied only for the Laughlin states and their variants [3,9], where fractional charge and anyon statistics are known to be closely related to each other [13]. Below we shall derive minimal ground-state degeneracy without assuming any relation between quasiparticle charge and statistics. A bonus of our reexamination is the identification of the discrete topological symmetry algebra that underlies the ground-state degeneracy, which can be used to classify topological orders that support Abelian anyonic quasiparticle excitations.

We will start with the braid group formalism [6,14,15] for fractional statistics. Consider for N quasiparticles in a toroidal system with size $L_x \times L_y$. The braid group generators [16] consist of σ_i ($i = 1, \cdots, N-1$), which exchanges the ith and $(i+1)$th particles clockwise without enclosing any other quasiparticle, and of τ_i and ρ_i ($i = 1, \cdots, N$), which represent moving the ith quasiparticle along a loop on the torus in x and y direction, respectively. (See Fig. 1.) Define operators $A_{i,j}$ and $C_{i,j}$ as $A_{j,i} = \tau_j^{-1}\rho_i\tau_j\rho_i^{-1}$ and $C_{j,i} = \rho_j^{-1}\tau_i\rho_j\tau_i^{-1}$, where $1 \le i < j \le N$. The exchange operators σ_i satisfy the following relations,

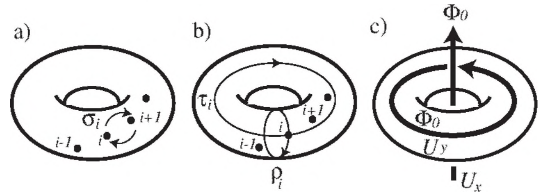

FIG. 1. (a) An exchange of quasiparticles; (b) Translations along nonshrinkable loops on the torus; (c) Two possible insertions of a unit flux Φ_0.

$$
\begin{gathered}
\sigma_k\sigma_l = \sigma_l\sigma_k, \qquad 1 \le k \le N-3, \qquad |l-k| \ge 2, \\
\sigma_k\sigma_{k+1}\sigma_k = \sigma_{k+1}\sigma_k\sigma_{k+1}, \qquad 1 \le k \le N-2, \\
\tau_{i+1} = \sigma_i^{-1}\tau_i\sigma_i^{-1}, \qquad \rho_{i+1} = \sigma_i\rho_i\sigma_i, \\
\tau_1\sigma_j = \sigma_j\tau_1, \qquad \rho_1\sigma_j = \sigma_j\rho_1, \qquad \sigma_i^2 = A_{i+1,i},
\end{gathered} \tag{1}
$$

where $1 \le i \le N-1$ and $2 \le j \le N-1$. And τ_i's and ρ_i's satisfy

$$
\begin{gathered}
A_{m,l}\tau_k = \tau_k A_{m,l}, \qquad A_{m,l}\rho_k = \rho_k A_{m,l}, \qquad \tau_i\tau_j = \tau_j\tau_i, \\
\rho_i\rho_j = \rho_j\rho_i, \qquad C_{j,i} = (\tau_i\tau_j)A_{j,i}^{-1}(\tau_j^{-1}\tau_i^{-1}), \\
A_{j,i} = (\rho_i\rho_j)C_{j,i}^{-1}(\rho_j^{-1}\rho_i^{-1}), \\
C_{j,i} = (A_{j,j-1}^{-1}\cdots A_{j,i+1}^{-1})A_{j,i}^{-1}(A_{j,i+1}\cdots A_{j,j-1}), \\
\tau_1\rho_1\tau_1^{-1}\rho_1^{-1} = A_{2,1}A_{3,1}\cdots A_{N-1,1}A_{N,1},
\end{gathered} \tag{2}
$$

where $1 \le k < l < m \le N$ and $1 \le i < j \le N$.

Let us now assume that the quasiparticles have a fractional charge $e^* = (p/q)e$, where p and q are mutually prime integers, and consider an adiabatic insertion of flux $2\pi/e$ through one of the holes of the torus. If the adiabatic flux insertion induces an infinitesimal electric field in the x direction, the process can be realized by a large gauge transformation U_x, in which the x component of the gauge field changes from $A_x = 0$ to $A_x = 2\pi/eL_x$. After the large gauge transformation, the gauge potential $A_x = 2\pi/eL_x$ will give rise to an Aharanov-Bohm phase $e^{-2\pi ip/q}$ when we apply τ_i. Therefore, we obtain

$$
U_x\tau_i = e^{-2\pi ip/q}\tau_i U_x. \tag{3}
$$

On the other hand, because σ_i and ρ_i do not encircle the flux, we have

$$
U_x\rho_i = \rho_i U_x, \qquad U_x\sigma_i = \sigma_i U_x. \tag{4}
$$

Similarly, using the adiabatic flux insertion, we can define another large gauge transformation U_y, in which the y component of the gauge potential changes from $A_y = 0$ to $A_y = 2\pi/eL_y$; and we have

$$
\begin{gathered}
U_y\tau_i = \tau_i U_y, \qquad U_y\rho_i = e^{-2\pi ip/q}\rho_i U_y, \\
U_y\sigma_i = \sigma_i U_y.
\end{gathered} \tag{5}
$$

We notice that the relations (3)–(5) are compatible with the braid group algebra (1) and (2). Thus the large gauge transformations U_a ($a = x, y$) are (outer) automorphism of the braid group operators:

$$
\sigma_i' = U_a\sigma_i U_a^{-1}, \qquad \tau_i' = U_a\tau_i U_a^{-1}, \qquad \rho_i' = U_a\rho_i U_a^{-1}. \tag{6}
$$

Namely, by using relations (3)–(5), one can check that the new operators σ_i', τ_i', and ρ_i' also satisfy the same braid group algebra as σ_i, τ_i, and ρ_i.

It is easy to verify that $U_xU_yU_x^{-1}U_y^{-1}$ commutes with all σ_i, τ_i, and ρ_i. Therefore, by Schur's lemma, for any irreducible representation, $U_xU_yU_x^{-1}U_y^{-1}$ is a (unimodular) c number; namely

$$
U_xU_y = e^{2\pi i\lambda}U_yU_x. \tag{7}
$$

As we will see, λ is rational and can be fixed by the requirement of a finite minimal ground-state degeneracy. It is a new many-body quantum number, also characterizing the topological order of the system, and is closely related to the fractional quantization of Hall conductance (see below).

Assume that the quasiparticles are *Abelian* anyons: $\sigma_j = e^{i\theta}\mathbf{1}$, where $\mathbf{1}$ is the unit matrix. Then the braid group representation is uniquely determined as

$$
\tau_j = e^{-2i\theta(j-1)}T_x, \qquad \rho_j = e^{2i\theta(j-1)}T_y, \tag{8}
$$

with T_x and T_y matrices satisfying

$$
T_xT_y = e^{-2i\theta}T_yT_x. \tag{9}
$$

On a torus we also have the constraint on N and θ, $e^{2iN\theta} = 1$. Assuming $N \ge 2$, θ/π must be a rational number, $\theta = \pi m/n$, where m and n are mutually prime integers. Thus, T_x and T_y satisfy

$$
T_xT_y = e^{-2\pi im/n}T_yT_x. \tag{10}
$$

The linear automorphisms induced by U_x and U_y now reduces to

$$
\begin{gathered}
U_xT_xU_x^{-1} = e^{-2\pi ip/q}T_x, \qquad U_yT_xU_y^{-1} = T_x, \\
U_xT_yU_x^{-1} = T_y, \qquad U_yT_yU_y^{-1} = e^{-2\pi ip/q}T_y.
\end{gathered} \tag{11}
$$

To count the ground-state degeneracy, we consider the following process. First create N pairs of quasiparticle and quasiholes out of the ground state, then move the ith quasiparticle by τ_i. After it returns to the original position, we pair annihilate all quasiparticles and quasiholes. This process defines an operation of τ_i to the ground state. Similarly, we define the operation of ρ_i and σ_i to the ground state. Throughout this Letter, we assume that the Fermi level lies in a gap and the gap remains finite in the operations above. Since for a system with Abelian anyonic excitations, the operations σ_i's on the ground state generate merely a phase. Thus we concentrate on the operations

$\tau_1 = T_x$ and $\rho_1 = T_y$ on the ground state; from Eq. (8) all other τ_i and ρ_i can be expressed in terms of them.

First, we reproduce the degeneracy due to the fractional statistics [6,16]. Let us take the basis of the ground state to be an eigenstate of T_x, $T_x|\eta\rangle = e^{i\eta}|\eta\rangle$. By applying T_y to $|\eta\rangle$ and using (10), the following new states are obtained: $T_x(T_y^s|\eta\rangle) = e^{i(\eta-2\pi sm/n)}T_y^s|\eta\rangle$, where s is an integer. Since the new states have n different eigenvalues of T_x, the ground state has n-fold degeneracy at least.

The charge fractionalization gives another constraint for the degeneracy. From Eq. (10), T_x and T_y^n commute with each other, $T_xT_y^n = T_y^nT_x$. Therefore, we can take the basis of the ground state which diagonalizes T_x and T_y^n simultaneously, $T_x|\eta_1, \eta_2\rangle = e^{i\eta_1}|\eta_1, \eta_2\rangle$, $T_y^n|\eta_1, \eta_2\rangle = e^{i\eta_2}|\eta_1, \eta_2\rangle$. By applying U_x and U_y to this and using Eq. (11), we have $T_x(U_x^sU_y^t|\eta_1, \eta_2\rangle) = e^{i(\eta_1+2\pi sp/q)}U_x^sU_y^t|\eta_1, \eta_2\rangle$ and $T_y^n(U_x^sU_y^t|\eta_1, \eta_2\rangle) = e^{i(\eta_2+2\pi tnp/q)}U_x^sU_y^t|\eta_1, \eta_2\rangle$ where s and t are integers. If $n/q = \mathcal{N}/\mathcal{Q}$ where $\mathcal{N}$ and $\mathcal{Q}$ are mutually prime integers, it is found that there are $q\mathcal{Q}$ sets of eigenvalues of T_x and T_y^n. This implies that the ground state has $q\mathcal{Q}$-fold degeneracy at least.

By combining the results above, we find that the minimal degeneracy of the ground state should be the least common multiplet of n and $q\mathcal{Q} = n\mathcal{Q}^2/\mathcal{N}$. Namely, the system has $n\mathcal{Q}^2$*-fold* ground-state degeneracy. This indicates clearly that the fractional statistics and the charge fractionalization are *both responsible* for the ground-state degeneracy. The minimal degeneracy obtained here includes both the results in Refs. [3,12] as special cases. More possibilities are predicted.

Up to now, we have not used noncommutating relation (7) between U_x and U_y. This relation contains an additional parameter λ, which is not fixed uniquely by e^* and θ. We believe this parameter λ could be determined by the low-energy effective Lagrangian, which we do not discuss here. In order for the degeneracy to be finite, λ has to be a rational number $\lambda = k/l$, where k and l are coprimes. At least in the following examples we find that the integers k and l can be determined by requiring the degeneracy be minimal given e^* and θ. In any case, the degeneracy is given by a multiple of $n\mathcal{Q}^2$.

Now we present some explicit representations of T_x, T_y, U_x, and U_y and corresponding degeneracy.

(1) *$\theta = \pi/n$ and $e^* = e/n$.*—This corresponds to the Laughlin state with $\nu = 1/n$. Because $\mathcal{N} = \mathcal{Q} = 1$, the minimum degeneracy is n. If we assume that U_x and U_y satisfy $U_xU_y = e^{-2\pi i/n}U_yU_x$, we can construct T_x, T_y, U_x, and U_y without increasing the degeneracy,

$$T_x = S_{n\times n}, \quad T_y = R_{n\times n}, \quad U_x = R_{n\times n}^{-1}, \quad U_y = S_{n\times n}. \tag{12}$$

Here $S_{n\times n} = \mathrm{diag}\{1, e^{i2\pi/n}, \cdots, e^{i2\pi(n-1)/n}\}$ and

$$R_{n\times n} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix}. \tag{13}$$

They satisfy $S_{n\times n}R_{n\times n} = e^{-2\pi i/n}R_{n\times n}S_{n\times n}$. This result reproduces the degeneracy given in Ref. [3].

(2) *$\theta = 0$ or $\theta = \pi$.*—The quasiparticles are bosons or fermions. Since $n = 1$, we obtain $\mathcal{N} = 1$ and $\mathcal{Q} = q$. Thus the minimal degeneracy is q^2 [12]. We find that if U_x and U_y commute with each other, the minimal degeneracy is realized as

$$\begin{aligned} T_x &= R_{q\times q} \otimes 1_{q\times q}, & T_y &= 1_{q\times q} \otimes R_{q\times q}, \\ U_x &= S_{q\times q}^p \otimes 1_{q\times q}, & U_y &= 1_{q\times q} \otimes S_{q\times q}^p. \end{aligned} \tag{14}$$

(3) *q and n are mutually prime.*—The degeneracy is nq^2. We can construct the following representation for T_x, T_y, U_x, and U_y:

$$\begin{aligned} T_x &= 1_{q\times q} \otimes R_{q\times q} \otimes S_{n\times n}^m, & T_y &= R_{q\times q} \otimes 1_{q\times q} \otimes R_{n\times n}, \\ U_x &= 1_{q\times q} \otimes S_{q\times q}^p \otimes 1_{n\times n}, & U_y &= S_{q\times q}^p \otimes 1_{q\times q} \otimes 1_{n\times n}, \end{aligned} \tag{15}$$

where the minimal degeneracy is realized. U_x and U_y commute with each other in this representation.

(4) *$n = \mathcal{N}q$ and $m = 1$.*—Because of $\mathcal{Q} = 1$, the minimum degeneracy is n. A representation is given by

$$T_x = S_{n\times n}, \quad T_y = R_{n\times n}, \quad U_x = R_{n\times n}^{-\mathcal{N}p}, \quad U_y = S_{n\times n}^{\mathcal{N}p}. \tag{16}$$

U_x and U_y satisfy $U_xU_y = e^{-2\pi i(\mathcal{N}p^2/q)}U_yU_x$.

(5) *$q = \mathcal{Q}n$.*—In this case, $\mathcal{N} = 1$, thus the least degeneracy is $n\mathcal{Q}^2$. When $\mathcal{Q}$ and n are mutually prime and $p = m = 1$, we can construct the following representation:

$$\begin{aligned} T_x &= S_{\mathcal{Q}\times\mathcal{Q}} \otimes S_{n\times n} \otimes 1_{\mathcal{Q}\times\mathcal{Q}}, \\ T_y &= 1_{\mathcal{Q}\times\mathcal{Q}} \otimes R_{n\times n} \otimes S_{\mathcal{Q}\times\mathcal{Q}}, \\ U_x &= R_{\mathcal{Q}\times\mathcal{Q}}^{-l} \otimes R_{n\times n}^{-k} \otimes 1_{\mathcal{Q}\times\mathcal{Q}}, \\ U_y &= 1_{\mathcal{Q}\times\mathcal{Q}} \otimes S_{n\times n}^{k} \otimes R_{\mathcal{Q}\times\mathcal{Q}}^{-l}, \end{aligned} \tag{17}$$

where $k/n + l/\mathcal{Q} = 1/\mathcal{Q}n$ (mod. 1); namely, the fractional part of the left side is equal to the right one. In this case, $U_xU_y = e^{-2\pi ik^2/n}U_yU_x$.

Here we would like to mention that the noncommutativity of the large gauge transformation $U_xU_y = e^{2\pi ik/l}U_yU_x$ is closely related to the topological order in the fractional quantum Hall effect. To see this, consider the degenerate ground states ϕ_K $(K = 1, \cdots, d)$, on a torus with boundary conditions parametrized by twisted phases θ and φ [4], satisfying $U_x|\theta, \varphi\rangle_K = |\theta + 2\pi, \varphi\rangle_K$ and $U_y|\theta, \varphi\rangle_K = |\theta, \varphi + 2\pi\rangle_K$. The Hall conductance is

$$\frac{e^2}{hd}\sum_{K=1}^{d}\int_0^{2\pi}\int_0^{2\pi}\frac{d\theta d\varphi}{2\pi i}\left[\left\langle\frac{\partial\phi_K}{\partial\varphi}\Big|\frac{\partial\phi_K}{\partial\theta}\right\rangle-(\theta\leftrightarrow\varphi)\right].$$

Because of $U_x^l U_y = U_y U_x^l$, we can take the basis which diagonalizes both U_x^l and U_y. In this basis a change in θ by $2\pi l$ or in φ by 2π leads the state back to itself. Therefore, we have a torus with $0 \le \theta < 2\pi l$ and $0 \le \varphi < 2\pi$. The above integral can be recast into

$$\frac{e^2}{hd}\sum_{r=0}^{d/l-1}\int_0^{2\pi l}\int_0^{2\pi}\frac{d\theta d\varphi}{2\pi i}\left[\left\langle\frac{\partial\phi_{rl+1}}{\partial\varphi}\Big|\frac{\partial\phi_{rl+1}}{\partial\theta}\right\rangle-(\theta\leftrightarrow\varphi)\right],$$

since the degenerate ground states ϕ_K satisfy $\phi_{rl+m}(\theta + 2\pi, \varphi) = \phi_{rl+m+1}(\theta, \varphi)$ $(r = 0, \cdots, d/l - 1, m = 1, \cdots, l - 1)$. Therefore we have the following formula for the Hall conductance, generalizing the result of Ref. [4]:

$$\sigma_{xy} = \frac{e^2}{hd}\sum_{r=0}^{d/l-1} I_r = \frac{e^2}{h}\frac{I}{l}. \tag{18}$$

Here the integer I_r is a generalization of the topological number (first discovered in Ref. [17]) defined by ϕ_{rl+1}. For the second equality, we have noted that all I_r's take the same value I, since the degenerate ground states are related to each other by symmetry operations T_x and T_y. This indicates clearly that the noncommutativity $\lambda = k/l$ of the large gauge transformations is essential to fractional quantization of the Hall conductance.

The fractional charge we discussed above can be any conserved $U(1)$ quantum number, with the flux threading understood as twisted boundary conditions. The minimal degeneracy obtained can be generalized to a high-genus Riemann surface: If the genus is g, for Abelian topological orders we find g commuting copies of the discrete algebra presented above, so the minimal degeneracy is $n^g Q^{2g}$.

To conclude, in this Letter we have proposed a discrete symmetry algebra, Eqs. (7), (10), and (11), of the operations T_x, T_y, U_x, and U_y, that completely characterizes the ground-state subspace of a generic *Abelian* topological order, i.e., that supports Abelian anyonic excitations, on a torus. The identification verifies the old idea that the ground-state degeneracy in a topological phase is due to the emergence of a discrete symmetry [2,5]. We note that the algebra identified is indeed of topological origin and contains only three fractional parameters: quasiparticle charge e^*/e, anyon statistics $\theta/2\pi$, and flux noncommutativity λ. Ground-state degeneracy is determined by the representations of this symmetry algebra.

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